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ORBIT OF AN EQUATORIAL SATELLITE OF THE EARTH

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The solution of the problem of motion of a point in the equatorial plane of an oblate ellipsoid is based on the existence of two integrals of motion for the case of any central force dependent on the distance. As a result of the above the problem is reduced to quadradtures [o] or subject to a direct qualitative analysis [2]. It is not devoid of interest to examine this solution by applying it to the concrete problem of the motion of an equatorial satellite of the Earth. The solution of this problem in polar coordinates is expressed in elliptical functions. Taking into account that it is practical to resolve the general problem of satellite motion in osculating elements [3, 4], it is useful to reveal the character of their variation in the case permitting the exact solution, so as to detect the link between the motion's properties and the behavior of the osculating elements.

Considered in the current paper are the properties of the Earth's equatorial satellite trajectories and the behavior of orbit's osculating elements. Certain main characteristics of the motion are brought out.

* Orbita ekvatorialnogo sputnika Zemli.

We have for the potential V with the precision to terms of the first order of smallness relative to Earth's ablation

$$V = \frac{\mu}{r} + \varepsilon \frac{R^2\mu}{3r^3}, \qquad (1)$$

where r is the distance from the satellite to the attracting center; ε is a dimensionless coefficient, which according to [5] we shall take as

$$\varepsilon = \alpha - \frac{m}{2}; \quad m = \frac{\Omega^2 R}{g_R},$$

and the numerical value $\varepsilon = 0.0016331$; R is the equatorial radius of the Earth, Ω is the angular velocity of Earth's rotation; g_R is the gravitational acceleration at the equator; α is the Earth's ablation with

$$\left(\alpha = \frac{R - R_\pi}{R},\right.$$

where R_π is the Earth's polar radius; $\mu = fM$, and f is the gravitational constant, M is the mass of the Earth.

The considered motion is plane and it will have the area integral

$$r^2 \frac{d\theta}{dt} = C \qquad (2).$$

and the energy integral

$$v^2 - \frac{2\mu}{r} - \varepsilon \frac{2\mu R^2}{3r^3} = h. \qquad (3)$$

Here

$$\left.\begin{aligned} h &= v_0^2 - \frac{2\mu}{r_0} - \varepsilon \frac{2\mu R^2}{3r_0^3}, \\ C &= r_0 v_0 \cdot \sin\varphi, \end{aligned}\right\} \qquad (3a)$$

where φ is the angle between the initial directions of the radius-vector **r** and the velocity vector $\boldsymbol{v}$, ϑ is the polar angle counted from a fixed direction.

The character of the motion is determined by the radicals of the polynomial

$$P(u) = u^3 - \frac{3C^3}{2\varepsilon\mu R^2} u^2 + \frac{3C^2}{\varepsilon R^2} u + \frac{3hC^3}{2\varepsilon\mu R^2}, \quad u = \frac{C}{r} \tag{4}$$

Assume that $h < 0$. Contrary to the case $h \geqslant 0$, such condition leads to trajectories that do not drift to infinity. In the case $h < 0$, the polynomial (4) has either one or three real positive radicals. To real initial velocities imparted to the Earth's satellite, corresponds the case of three true radicals $0 < u_3 < u_2 < u_1$, and in real motion ($P(u) > 0$) u assumes a value in the intervals $u_3 \leqslant u \leqslant u_2$ or $u_1 \leqslant u \leqslant \infty$. . To the second interval corresponds a motion passing through the attraction center. At the initial data, corresponding to real conditions, the variation of u in the first of the indicated intervals corresponds to the motion of the Earth's artificial satellite.

The radicals u_1, u_2, u_3 may be expressed through the coefficients of the polynomial (4) as a result of the solution of the cubic equation $P(u) = 0$. Then the integration of the motion equations leads to the trajectory equation of the equatorial satellite :

$$r(\theta) = \frac{\dfrac{C}{u_3}}{1 + \left(\dfrac{u_2}{u_3} - 1\right) \operatorname{sn}^2 [(\theta - \theta_1)\sqrt{(u_1 - u_3)A},\ k^2]}, \tag{5}$$

$$k^2 = \frac{u_2 - u_3}{u_1 - u_3} < 1, \quad A = \frac{\varepsilon\mu R^2}{6C^3}. \tag{6}$$

The function $sn^2\chi$ has a period 2K along the argument χ,
where K is the total elliptical integral

$$K = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}\,.$$

Consequently, the radius-vector is a periodical function of the argument θ with a period T_r:

Since $T_r \neq 2\pi$, the distance to the satellite will not be equal to the initial one, at radius-vector's rotation by 2π around the center of attraction. The equality to the initial distance is reached at rotation by an angle T_r. This means that the trajectory is not closed generally-speaking, but has a form schematically indicated in Fig.1.

We may see from (5) that at points

$$\theta_{max} = \theta_1 \text{ and } \theta_{min} = \theta_1 + \sqrt{\frac{1}{A(u_1-u_3)}}\,2K$$

r reaches its correspondingly greatest and smallest values

$$r_\alpha = \frac{C}{u_3}\,, \quad r_\pi = \frac{C}{u_2}\,.$$

Therefore, u_2 and u_3 have the following sense :

$$u_2 = \frac{C}{r_\pi}\,, \quad u_3 = \frac{C}{r_\alpha}\,,$$

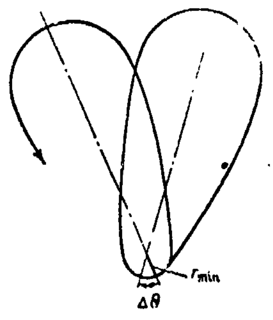

Fig.1. Scheme of the trajectory of an equatorial satellite.

where r_α and r_π are respectively the apogee and perigee distances. The quantities r_α and r_π may be considered as sufficiently accurately known on the basis of the theory of satellite motion along elliptical orbits, thus not requiring to resort to cumbersome formulae corresponding to the solution of the cubic equation $P(u) = 0$. We thus shall consider in the following r_α and r_π as precisely known.

Introducing the parameters

$$\tilde{p} = \frac{2r_\pi r_\alpha}{r_\pi + r_\alpha}, \quad \tilde{e} = \frac{r_\alpha - r_\pi}{r_\alpha + r_\pi}, \quad \alpha = \sqrt{(u_1 - u_3) A}$$

and on condition of counting the motion from the perigee (in perigee $\theta = \theta_\pi = 0$), we may write the trajectory equation in the form

$$r(\theta) = \frac{\tilde{p}}{1 + \tilde{e}\,[\mathrm{cn}^2 \psi - \mathrm{sn}^2 \psi]}; \quad \psi = \alpha\theta. \tag{5a}$$

At limit, for $\varepsilon \to 0$, $\mathrm{cn}\psi = \cos\frac{\theta}{2}$, $\mathrm{sn}\,\psi = \sin\frac{\theta}{2}$ and (5a) transforms into an elliptical trajectory.

During the time elapsed between two consecutive passings of the perigee by the satellite, the perigee's radius-vector will rotate by the angle

$$\Delta\theta = T_r - 2\pi.$$

Let us determine the dependence of $\Delta\theta$ on orbit's parameters. Since according to one of the Vieta formulae

$$u_1 + u_2 + u_3 = \frac{3C^2}{2\varepsilon\mu R^2} = \frac{1}{4A},$$

we must have

$$\Delta\theta = \frac{4K(k^2)}{\sqrt{1 - 4A(u_2 + 2u_3)}} - 2\pi, \tag{8}$$

where, by the strength of (6)

$$k^2 = \frac{4A(u_2 - u_3)}{1 - 4A(u_2 + 2u_3)}. \tag{9}$$

In case of absence of perturbations A = 0 and we obtain from (8) $\Delta\theta = 0$. In the general case $\Delta\theta \neq 0$.

Let us express (8) through $\tilde{p}$ and $\tilde{e}$. We have the following correlations :

../..

$$1 - 4A\,(u_2 + 2u_3) = 1 - \frac{2}{3}\,\varepsilon\,\frac{R^2}{\widetilde{p}^2}\,(3 - \widetilde{e}),$$
$$4A\,(u_2 - u_3) = \frac{4}{3}\,\varepsilon\,\frac{R^2}{\widetilde{p}^2}\,\widetilde{e}. \qquad (9a)$$

Since besides that

$$\widetilde{p} = r_\pi\,(1 + \widetilde{e}),$$

formulac (8) – (10) provide the dependence $\Delta\theta$ on orbit parameters serached for :

$$\Delta\theta = f\,(r_\pi, \widetilde{e}).$$

This dependence is plotted in Fig. 2 for $r_\pi = R + h_\pi$ with $h = 320$ km. We may see that the drift velocity of the perigee does not exceed 0.6° per revolution. For an orbit with a greater perigee height the drift velocity will be lower. We may see also that the perigee drift velocity depends on orbit prolateness $\widetilde{e}$ almost linearly.

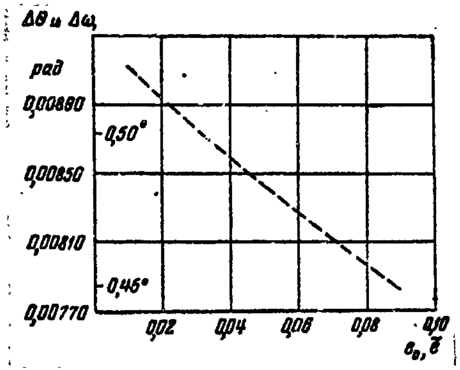

Fig. 2. Secular course of the perigee of equatoral satellite's orbit ($h_\pi = 320$ k,).

The curvature $\varkappa_\alpha$ of the trajectory at apogee (i. e. at $\theta = \theta_1$) is not equal to the curvature $\varkappa_\pi$ at perigee (i. e. at $\theta = \theta_{min}$) :

$$\varkappa_\pi = \frac{1}{\widetilde{p}}\,[1 + \widetilde{e} - 4\widetilde{e}\alpha^2(1 - k^2)],$$
$$\varkappa_\alpha = \frac{1}{\widetilde{p}}\,[1 - \widetilde{e} + 4\widetilde{e}\alpha^2],$$

$$\alpha^2 = \frac{1}{4}\left\{1 - \frac{2}{3}\,\varepsilon\,\frac{R^2}{\widetilde{p}^2}\,(3 - \widetilde{e})\right\}, \qquad k^2 = \frac{4\varepsilon\,\frac{R^2}{\widetilde{p}^2}\,\widetilde{e}}{3 - 2\varepsilon\,\frac{R^2}{\widetilde{p}^2}\,(3 - \widetilde{e})}.$$

It is easy to show that always takes place. Consequently, the trajectory may be represented as some oval form, rotating

in its own plane, sharper at perigee, less tapered at apogee (see Fig. 1).

Knowing the satellite's trajectory, we may determine the dependence on the motion on time by integrating (2).

The problem here considered may also be solved by the standard celestial mechanics' method of osculating elements.

For the case of a central perturbing force the equations in osculating elements have the form [6] :

$$\left\{\begin{aligned} &\frac{de}{dt} = S_1 \sin\vartheta && (11)\\ &\frac{d\omega}{dt} = -\frac{1}{e} S_1 \cos\vartheta, && (12)\\ &\frac{dp}{dt} = 0. && (13)\end{aligned}\right.$$

Here $S_1 = \sqrt{\frac{p}{\mu}} S$, a $S(r)$ is the acceleration of the perturbing force, e — the eccentricity of the osculating ellipse, p is its focal parameter. The absolute value of orbit's radius-vector r is given by the formula

The angle ϑ is counted starting from the direction toward orbit perigee. This direction is not fixed in space, but constitutes a variable angle ω with a certain fixed direction.

By the strength of perturbation centrality, the equations for the determination of the inclination angle i of the orbit to the equator and of the longitude Ω of orbit's ascending node do not enter into the equations of osculating elements, for the angle i remains constant ($i = 0$), while the motion of the node adds up with orbit perigee motion into a total effect of orbit rotation in its plane, described by the equation (12).

From (13) we have $p = p_0$, i.e. the focal parameter of the equatorial orbit remains constant. This property is revealed at any

central perturbations [6] and it characterizes to a certain degree the constancy of the orbit's shape in the rotating system of coordinates.

Let us pass in the equations (11) and (12) from the time argument t to the argument of true anomaly ϑ.

In the general case the transition formula is given by [3] and [7] for example. In the considered case, the link between dt and $d\vartheta$ may be obtained in the following manner: In case of central forces there exists an area integral

$$r^2 \frac{d\theta}{dt} = C = \sqrt{\mu p}.$$

But since the polar angle $\theta = \vartheta + \omega$, $\frac{d\vartheta}{dt} = \frac{C}{r^2} - \frac{d\omega}{dt}$. According to (12), we have

Consequently,

$$\frac{d\vartheta}{dt} = \frac{\sqrt{\mu p}}{r^2} + \frac{1}{e}\sqrt{\frac{p}{\mu}}\, S\cos\vartheta,$$

$$dt = \frac{r^2 d\vartheta}{\sqrt{\mu p} + \frac{r^2}{e}\sqrt{\frac{p}{\mu}}\, S\cos\vartheta}. \tag{14}$$

The equations (14) thus give give the link between the differentials dt and $d\vartheta$. Let us note that ϑ may also be a nonmonotonic function of time. That is why the variable ϑ is not always practical as an argument.

For the case of a centrally perturbing acceleration $S(r)$, the **equations** in osculating elements take the form

$$\frac{d\omega}{d\vartheta} = \frac{-r^2 S\sqrt{\frac{p}{\mu}}\cos\vartheta}{\sqrt{\mu p}\cdot e + \sqrt{\frac{p}{\mu}}\, r^2 S(r)\cos\vartheta}, \tag{15}$$

$$\frac{de}{d\vartheta} = \frac{\sqrt{\frac{p}{\mu}}\, r^2 S\cdot e\sin\vartheta}{\sqrt{\mu p}\cdot e + \sqrt{\frac{p}{\mu}}\, r^2 S(r)\cos\vartheta}. \tag{16}$$

At the same time it is assumed that $\frac{d\vartheta}{dt} \neq 0$.

The solution of equations (15) - (16) for any S (r) is reduced to quadratures. Let us designate

$$r^2 S(r) = f(r) \equiv F(1 + e\cos\vartheta),$$
$$e = y,$$
$$e\cos\vartheta = z.$$

Then the integral of equation (16) will be written :

$$y^2 + \frac{2}{\mu}\int F(1+z)\,dz = \text{const.} \qquad (17)$$

Hence, e is determined as a function of ϑ; after that by taking the quadrature from (15) $\omega = \omega(\vartheta)$ is determined. In the particular case of satellite motion in the equatorial plane of the Earth we have

$$S = -\frac{\varepsilon\mu R^2}{r^4}.$$

Then, for the determination of e (ϑ) the following equation is given by (17) :

$$\frac{3}{2}(e^2 - e_0^2) = \varepsilon\frac{R^2}{p_0^2}\{(1+e\cos\vartheta)^3 - (1+e_0)^3\}; \quad e_0 = e(0). \qquad (18)$$

Hence, determining e (ϑ), we have the following formula for perigee drift :

$$\omega - \omega_0 = \int \frac{\varepsilon\frac{R^2}{p_0^2}(1+e\cos\vartheta)^2\cos\vartheta\,d\vartheta}{e - \varepsilon\frac{R^2}{p_0^2}(1+e\cos\vartheta)^2\cos\vartheta}. \qquad (19)$$

To determine perigee drift $\Delta\omega$ for one revolution (19) must be integrated from 0 to 2π. The dependence $\Delta\omega(e_0)$ so computed is plotted in Fig. 2. In the admitted scale of this Figure, the plotted dependences $\Delta\theta(\bar{e})$ and $\Delta\omega(e_0)$ practically coincide. Formulae (8) and (19) give the equivalent results with a precision to :

Presented are in Figures 3 and 4 the dependences of oscultaing elements on ϑ for various initial data.

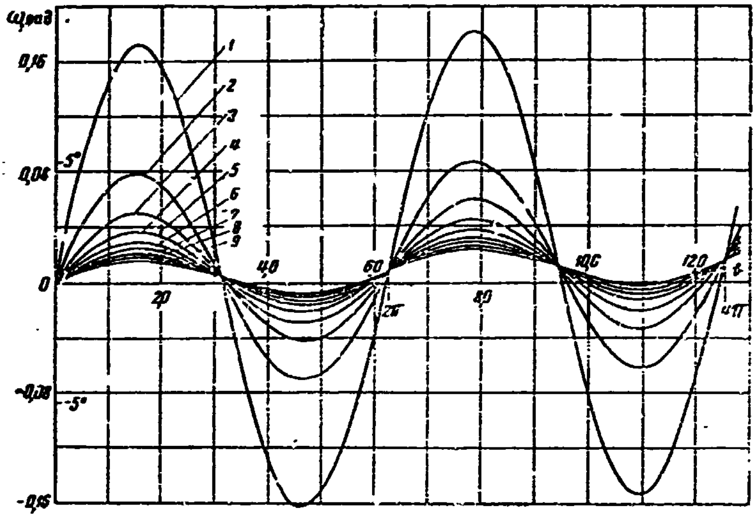

Fig. 3. Behavior of the osculating longitude of orbit perigee (for the case h = 320 km)

1.- initial value of eccentricity $e_0 = 0.01$; 2.- $e_0 = 0.02$; 3.- $e_0 = 0.03$ ... etc... 9.- $e_0 = 0.09$.

Both, the periodical as well as the secular perigee longitude drifts ω are equally well visible in Fig. 3. For small eccentricities the periodical oscillations ω are rather large (at $e_0 = 0.01$ the amplitude of oscillations is about 10°).

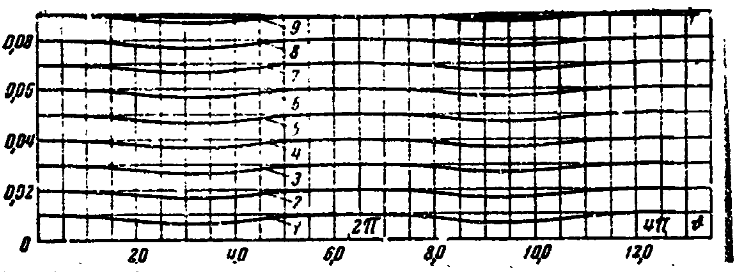

Fig. 4. Behavior of the osculating eccentricity of the orbit. (Same captions as those of Fig. 3).

The oscillations of the eccentricity e are rather small and they virtually do not depend on initial values e_0 (see Fig. 4). It follows from (18) that at $\vartheta = 0$, the eccentricity $e = e_{max} = e_0$, and at $\vartheta = \pi$ we have

if only e_0 is sufficiently great so that $e_{min} > 0$. The fact that the osculating eccentricity is smaller at the apogee than at the perigee implies that the curvature of the trajectory in apogee is lesser than in perigee, as this was already shown. For very small e_0 (comparable in magnitude with ε or even still smaller) the character of $e(\vartheta)$ dependence varies in comparison with that plotted in Fig. 4, while $e(\vartheta) > 0$ is always obtained.

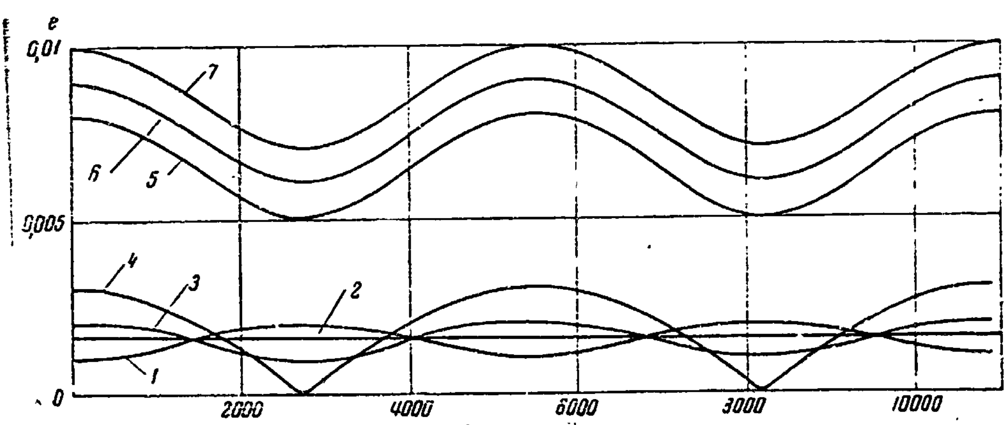

Fig. 5. Behavior of orbit's osculating eccentricity at low initial values of eccentricity:

1. – $e_0 = 0.001$; 2 – $e_0 = 0.0016$; 3. – $e_0 = 0.002$; 4. – $e_0 = 0.003$; 5. – $e_0 = 0.008$; 6. – $e_0 = 0.009$; 7. – $e_0 = 0.010$.

Plotted is in Fig. 5 the dependence $e(t)$ for small e_0. It must be noted that at small the dependence ϑ on time becomes nonmonotonic, and, as already shown, ϑ becomes impracticable as an independent variable. In particular, for a circular orbit $\vartheta =$ const. and cannot serve as an independent variable.

As a matter of fact, equations (11), (12) and (14) have a solution $\vartheta = 0$, $e = e_o = \varepsilon \frac{R^2}{r_o^2}$, $p = p_o$, $r = r_o$ and

This solution shows that a circular orbit in osculating elements is circumscribed by an ellipse rotating with an angular velocity

$$T_r = \frac{2}{\sqrt{(u_1 - u_3) A}} K. \qquad (7)$$

while the satellite is always in the perigee of that ellipse ($\vartheta = o$)*

We thus see that in a perturbed motion $\frac{d\vartheta}{dt} = 0$ is verified for as small disturbances as may be desired ($S \neq 0$). This points to the principal value of the transition formula (14) and the corresponding formula for the spatial motion. Meantime, the principalness of precise transition formulae from the time argument to another argument as an independent value is diregarded in certain works. Thus, in the paper [8] it is asserted that the equation taking into account the precise transitiom from time to the true anomaly is preferable for great perturbations. In fact we see that an inaccurate substitution of variables may lead to substantial errors in the consideration of osculating elements of a perturbed motion even for as small perturbations as one may wish.

**** THE END ****

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* This example belongs to T. M. Eneyev.

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